Theoretical Correlation of Velocity and Eddy Viscosity for Flow Close to a Pipe Wall

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There are available in the literature (*1, 2, 6, 9, 10, 11, 14, 15, 16, 19, 20*) several empirical expressions for the velocity and eddy viscosity distributions in the vicinity of a pipe wall. These expressions represent the data on velocity and turbulent shear stress in pipes fairly well, but none of these satisfies the equations of mean motion in the vicinity of the wall. Recently Tien and Wasan (*17*) attempted to establish compatible formulae for the distributions of velocity and turbulent shear stress in the wall region for two-dimensional turbulent channel flow. But the accuracy of the predicted result is not completely known because of the lack of experimental data on eddy viscosity distribution in two-dimensional channel flow. In this note, with the equations of mean motion, the authors present compatible expressions for the velocity and continuous eddy viscosity distributions near a pipe wall.

The equations of continuity and momentum are used as a starting point. The continuity equation for the fluctuating component velocities and the equations of mean motion for fully developed turbulent incompressible flow can be written in cylindrical coordinates as

$$\frac{\partial u}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}(rv) + \frac{1}{r}\frac{\partial w}{\partial \phi} = 0 \quad (1)$$

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = -\frac{1}{r}\frac{d}{dr}(r\,\overline{uv}) + \nu\left(\frac{d^2U}{dr^2} + \frac{1}{r}\frac{dU}{dr}\right) \quad (2)$$

$$\frac{1}{\rho}\frac{\partial p}{\partial r} = -\frac{1}{r}\frac{d}{dr}(r\overline{v^2}) + \frac{\overline{w^2}}{r} \quad (3)$$

and

$$\overline{vw} = 0 \quad (4)$$

For fully developed turbulent pipe flow the conditions are $V = W = 0$, and the mean velocity is independent of x.

Near the wall, the fluctuating and the mean velocities can be expressed in Taylor series

$$u(x, y, \phi, t) = u_0 + u_1 y + u_2 y^2 + ---- \quad (5)$$

$$v(x, y, \phi, t) = v_0 + v_1 y + v_2 y^2 + ---- \quad (6)$$

$$w(x, y, \phi, t) = w_0 + w_1 y + w_2 y^2 + ---- \quad (7)$$

and

$$U = U_0 + U_1 y + U_2 y^2 + U_3 y^3 + U_4 y^4 + ---- \quad (8)$$

where

$$Q_n = \frac{1}{n!}\left(\frac{\partial^n Q}{\partial y^n}\right)_{y=0}; \quad (Q = u, v, w, \text{ or } U)$$

and y is the distance from the pipe wall. Since velocities vanish at the wall $u = v = w = U_0 = 0$. Also Equation (1) requires that $v_1 = 0$. Combination of Equations (5) and (6) gives

$$\overline{uv} = \overline{u_1 v_2}\, y^3 + (\overline{u_2 v_2} + \overline{u_1 v_3})\, y^4 + ---- \quad (9)$$

The above result demonstrates that the Reynolds stress $\overline{uv}$ vanishes near the pipe wall with a power of the distance y not less than three. This is in agreement with the result derived by Townsend (*18*) and by Tien and Wasan (*17*) for fully developed turbulent flow in a two-dimensional channel, with Reichardt's (*11*) empirical result

ABSTRACTS AND KEY WORDS

The thermodynamic behavior of the real ideal gas, Martin, Joseph J., Chem. Eng. Progr. Symposium Ser. No. 44, 59, p. 120 (1963).

Key Words: Real Ideal Gas-8, Second Virial Coefficient-8, Residual Volume-8, Thermodynamic Behavior-9, Thermodynamic Derivations-10.

Abstract: Thermodynamic relations for real gas behavior at low pressure are given. Effect of pressure, volume, or density on internal energy, enthalpy, heat capacity, and Joule-Thomson coefficient is considered. Interrelation of second virial coefficient, residual volume, and slope of compressibility lines is shown, and generalized second virial coefficient equation presented.

The P-V-T behavior of selected hydrogen-methane and hydrogen-ethane mixtures, Solbrig, Charles W., and Rex T. Ellington, Chem. Eng. Progr. Symposium Ser. No. 44, 59, p. 127 (1963).

Key Words: Refinery Gas-1, Hydrogen-Hydrocarbon Mixtures-1, Hydrogasification Gas-1, Pipeline Gas-2, Hydrogen-2, Hydrogen-3, Pressure-6, Temperature-6, Composition-6, Density-7, Compressibility Factor-7, P-V-T Behavior-8, Enthalpy-9. B. Hydrogen-Hydrocarbon Mixtures-1, Methane-2, Ethane-2, Hydrogen-2, Hydrogen-3, Pressure-6, Temperature-6, Composition-6, Density-7, Compressibility Factor-7, P-V-T Behavior-8, Compressibility Apparatus-10, P-V-T Apparatus-10.

Abstract: Hydrogasification reactor off gas and refinery gases are hydrogen-hydrocarbon mixtures. Cryogenic temperatures are used for separation. P-V-T data are presented for a 90% hydrogen-10% methane mixture for −210° to 300°F. and 20, 50, 80 and 90% ethane with hydrogen for 70° to 300°F., all for pressures to 3,000 lb./sq. in. abs. to facilitate compression and heat exchange calculations.

in smooth pipes, and with the hypothesis of Lin, Moulton and Putnam (*6*) who implicitly assumed a third power on y in order to calculate mass transfer data in turbulent gaseous and liquid streams.

Since the quantities like $\overline{u_1 v_2}$ are not experimentally measurable, they are to be determined by using equations of mean motion. Differentiation of Equation (3) with respect to x gives $\frac{\partial^2 p}{\partial x \partial r} = 0$; hence, $\frac{\partial p}{\partial x}$ is independent of r and Equation (2) readily integrates to

$$\frac{r^2}{2} \frac{1}{\rho} \frac{\partial p}{\partial x} = - r\left(\overline{uv} - \nu \frac{dU}{dr}\right) + C(x) \quad (10)$$

As r approaches zero all the terms of the above Equation approach zero. Then Equation (10) gives

$$C(x) = 0 \quad (11)$$

Combining Equations (8), (10), and (11), the expression for the turbulent shear stress in the vicinity of the wall becomes

$$\overline{uv} = -\nu \left[\left(\frac{1}{2\rho\nu} \frac{\partial p}{\partial x} R + U_1 \right) + \left(-\frac{1}{2\rho\nu} \frac{\partial p}{\partial x} + 2U_2 \right) y + 3U_3 y^2 + 4U_4 y^3 + 5U_5 y^4 + ---- \right] \quad (12)$$

As the first three coefficients of the above expression must vanish, therefore, from Equation (9) it is clear that

$$U_1 + 2RU_2 = 0$$
$$U_3 = 0$$
$$\overline{u_1 v_2} = -4\nu U_4$$
$$\overline{u_2 v_2} + \overline{u_1 v_3} = -5U_5\nu, \text{etc.}$$

Therefore, the mean velocity and the turbulent shear stress near the wall can be expressed as

$$U = U_1 y - \frac{1}{2R} U_1 y^2 + U_4 y^4 + U_5 y^5 + ----$$

and

$$\overline{uv} = -4\nu U_4 y^3 - 5\nu U_5 y^4 ----$$

In dimensionless form these distributions near the wall become

$$U^+ = \left(1 - \frac{y}{2R}\right) y^+ + U_4^+ (y^+)^4 + U_5^+ (y^+)^5 + ---- \quad (13)$$

$$\overline{uv}^+ = -4U_4^+ y^{+3} - 5U_5^+ y^{+4} ---- \quad (14)$$

where $U^+ = U/U_\tau$

and $\overline{uv^+} = \overline{uv}/U_\tau^2$

According to Hinze (*4*), the maximum value of the term $\left(\frac{y}{2R}\right)$ in Equations (13) varies from 0.1 to 0.0008 for a Reynolds number range of 5×10^3 to 10^6; therefore this term can be neglected compared to unity. Also, the series expression of U^+ is truncated up to the fifth order and the compatible expression for $\overline{uv^+}$ is up to the fourth order. Equation (13) can now be expressed as

$$U^+ = y^+ + U_4^+ (y^+)^4 + U_5^+ (y^+)^5 \quad (15)$$

In the wall region the shear stress can be considered constant and the flow is determined by the wall shear stress, by the fluid viscosity, and by the distance from the wall. Therefore according to the wall similarity concept, the Reynolds stress $\overline{uv^+}$ should be function of y^+ only. Hence, coefficients U_4^+ and U_5^+ are universal constants. To calculate the coefficients U_4^+, U_5^+, and the value of y^+ at which a smooth and continuous transition to the logarithmic distribution occurs, the values of U^+ and the first and second derivative of U^+ with respect to y^+ given by Equation (15) are matched with corresponding values given by von Kármán's (*20*) logarithmic distribution of the form

$$U^+ = 2.5 \ln y^+ + 5.5 \quad (16)$$

The transition is found to occur at about $y^+ = 20$ and the velocity distribution for $y^+ \leqq 20$ is obtained as

$$U^+ = y^+ - 1.04 \times 10^{-4} (y^+)^4 + 3.03 \times 10^{-6} (y^+)^5 \quad (17)$$

From Equation (14) the expression for the turbulent shear stress distribution becomes for $y^+ \leqq 20$

$$\overline{uv^+} = 4.16 \times 10^{-4} (y^+)^3 - 15.15 \times 10^{-6} (y^+)^4 \quad (18)$$

When Equations (17) and (18) are combined, the ratio of eddy viscosity to kinematic viscosity near the pipe wall is given by

$$\frac{\epsilon}{\nu} = \frac{\overline{uv^+}}{\frac{dU^+}{dy^+}} = \frac{4.16 \times 10^{-4} (y^+)^3 - 15.15 \times 10^{-6} (y^+)^4}{1 - 4.16 \times 10^{-4} (y^+)^3 + 15.15 \times 10^{-6} (y^+)^4} \quad (19)$$

Equation (19) gives the eddy viscosity distribution in the viscous sublayer and the buffer region of the turbulent velocity profile, that is, for $y^+ \leqq 20$.

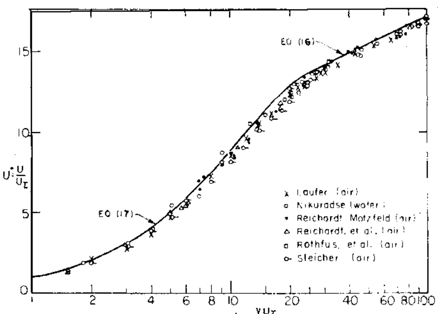

Fig. 1. Mean velocity distribution near the pipe wall.

In Figure 1 the velocity distribution as computed from Equation (17) is compared with the existing data of Laufer (*5*), Nikuradse (*7*), Reichardt et al. (*12*), Rothfus et al. (*13*), and Sleicher (*14*) for turbulent flow of fluids in pipes. The proposed distribution satisfies the equations of mean motion and gives a smooth and continuous transition to the logarithmic distribution in the turbulent core.

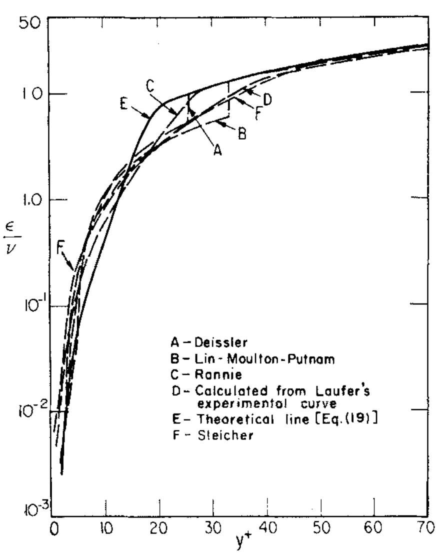

Fig. 2. Eddy viscosity distribution near the pipe wall.

In Figure 2 the eddy viscosity distribution as given by Equation (19) is compared with the empirical expressions of Deissler (*1*), Lin, Moulton, and Putnam (*6*), Rannie (*10*), and Sleicher (*14*). A significant result is that the proposed distribution of ϵ/ν is continuous. This result agrees with the experimental data of Laufer which indicates that the degree of turbulence in the moving fluid varies continuously from the wall to the axis of a pipe.

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NOTATION

f = functional notation
p = mean pressure at any point
r = radial distance measured from axis of tube
R = radius of tube
t = time
u,v,w = instantaneous velocities fluctuations in x, r, and ϕ directions, respectively
U = mean velocity at any point
U_τ = friction velocity; $U_\tau^2 = -\nu\left(\frac{dU}{dr}\right)_{r=R}$
V,W = mean velocities in radial and azimuthal directions, respectively
x = axial direction
y = distance measured from the wall
y^+ = yU_τ/ν dimensionless distance
ν = kinematic viscosity
ϵ = eddy viscosity
ρ = density of fluid
ϕ = azimuthal direction

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Nonuniform Residence Times and the Production of Intermediates in Tubular Reactors

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The assumption of plug, or uniform, flow and the concomitant assumption of a uniform residence time is a frequent assumption in reactor engineering. This assumption may lead to gross errors, particularly if the product of interest is an intermediate in the reaction scheme. To illustrate this, consider the simple sequence

$$A \xrightarrow{k_1} R \xrightarrow{k_2} S \qquad (1)$$

and assume that R is the desired product. In this case

$$Y = \left[\frac{C_R}{C_{A_0}\left(\frac{k_1}{k_2 - k_1}\right)}\right] = e^{-k_1\theta} - e^{-k_2\theta} \qquad (2)$$

The value of the modified dimensionless concentration ratio, Y, becomes the plug flow value, Y_{pf}, when the contact time, θ, is taken as constant for all fluid elements passing through the reactor. If the distribution-of-residence times of fluid elements leaving the reactor is known as a function of position at the reactor exit, that is

$$\theta = f(\sigma) \qquad (3)$$

then Equations (2) and (3) may be combined to yield a mean value of the dimensionless concentration ratio across the reactor exit, Y_m.

For example, in a tubular reactor the velocity profile may be written

$$v = B\,(1 - \sigma^n) \qquad (4)$$

wherein B is chosen so that

$$\frac{V}{A} = v_m = 2B\int_0^1 (1 - \sigma^n)\sigma d\sigma \qquad (5)$$

regardless of the value of the exponent, n. From Equation (5) one finds

for $n = 2$, $B = 2\ v_m$ (the parabolic profile)
$n = 4$, $B = 1.5\ v_m$
$n = 8$, $B = 1.25 v_m$

Thus

$$\frac{Y_m(\sigma_c)}{Y_{pf}(\sigma_c)} = \frac{2\int_0^{\sigma_c}\sigma\left[e^{-\frac{K_1}{1-\sigma^n}} - e^{-\frac{K_2}{1-\sigma^n}}\right]d\sigma}{\sigma_c^2\left[e^{-K_1\frac{B}{v_m}} - e^{-K_2\frac{B}{v_m}}\right]} \qquad (6)$$

Equation (6) presents the ratio of the actual mean concentration of R in a tubular element about the axis having diameter $2\sigma_c$ to the concentration which would exist in this tubular element if the flow in it were uniform at a velocity equal to the integral mean value in the entire tube. The equation may be solved to examine the effect of the plug flow assumption if values of K_1, K_2, and n are available. This has been done for several sets of these constants, chosen for purely illustrative purposes. The results are shown on Figures 1 and 2. The former indicates the effect of sampling across the tube through various axisymmetric element sizes, and the latter illustrates the effect on the average concentration issuing from the entire tube cross section